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Math 3034 HW – Problem Set 17

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5.3.2. Explain why each of the following is not a function.

(a) $\alpha : \mathbf{R} \to \mathbf{R}$ defined by

$$\alpha(x) = \frac{x}{x^2 - 4}$$

for every $x \in \mathbf{R}$.

There exists an x that is not mapped by α, specifically x=2 and x=-2.

(b) $\beta : \mathbf{R} \to \mathbf{R}$ defined by $\beta(x) = x \ln |x|$ for every $x \in \mathbf{R}$.

There exists an x that is not mapped by β, x=0.

(c) $\gamma : \mathbf{Q} \to \mathbf{Q}$ defined as follows: For a rational number r, write $r = a/b$, where a and b are integers and $b \neq 0$. Set

$$\gamma(r) = \gamma(\frac{a}{b}) = \frac{a+b}{a^2+b^2}.$$

This is not a function because whereas 2a/2b is mathematically the same as a/b, $\gamma\left(\frac{2a}{2b}\right) \neq \gamma\left(\frac{a}{b}\right)$. To see this,

$$\gamma\left(\frac{a}{b}\right) = \frac{a+b}{a^2+b^2}$$

and

$$\gamma\left(\frac{2a}{2b}\right) = \frac{2a+2b}{4a^2+4b^2} = \frac{a+b}{2(a^2+b^2)}$$

so

$$\gamma\left(\frac{a}{b}\right) = .5\gamma\left(\frac{2a}{2b}\right)$$

∴ they are not equal.

(d) $\lambda : \mathbf{Z}_8 \times \mathbf{Z}_8 \to \mathbf{Z}_6$ defined by $\lambda([a],[b]) = [ab]$ for all $([a],[b]) \in \mathbf{Z}_8 \times \mathbf{Z}_8$.
(NOTE: On the left, $[a]$ and $[b]$ are congruence classes mod 8, whereas on the right, $[ab]$ is a congruence class mod 6.)

Suppose that ([2],[4]) are ([a],[b]) in $\mathbb{Z}_8 \times \mathbb{Z}_8$. Then [ab] = [8]. In $\mathbb{Z}_8 \times \mathbb{Z}_8$, [8] is equivalent to [0] but in $\mathbb{Z}_6$, [8] is equivalent to [2], and $2 \neq 0$.

5.3.3. Let m and n be positive integers such that m divides n. Prove that $\alpha : \mathbf{Z}_n \to \mathbf{Z}_m$ defined by $\alpha([a]_n) = [a]_m$ is well-defined.

Proof:

Show that if $x_1, x_2 \in [a]_n$ and $x_1 = x_2$ then $\alpha(x_1) = \alpha(x_2)$. Let $x_1, x_2 \in [a]_n$ and assume that $x_1 = x_2$. Then, $(x_1 - a)|n$ and $(x_2 - a)|n$. Since m|n, we can say that m must also divide $(x_1 - a)$ and $(x_2 - a)$. Therefore, $(x_1 - a)|m$ and $(x_2 - a)|m$ means that $x_1, x_2 \in [a]_m$ and they are therefore mapped such that $\alpha(x_1) = \alpha(x_2)$, as claimed.//

5.3.5. Define $\beta : \mathbf{R} \to \mathbf{R}$ by $\beta(x) = 3x^2 + 5$ for every $x \in \mathbf{R}$. Prove that β is neither $1 - 1$ nor onto.

Proof: β is not one to one.

$\exists x_1, x_2 \in \mathbf{R}$ such that $x_1 \neq x_2$ and $\alpha(x_1) = \alpha(x_2)$. Picking $x_1 = -1$ and $x_2 = 1$, $x_1 \neq x_2$, and $\alpha(-1) = 8 = \alpha(1)$, as claimed.//

Proof: β is not onto.

$\exists y \in \mathbf{R}\ \forall x \in \mathbf{R}\ y \neq 3x^2+5$. Picking y to be 0, it is clear that $3x^2+5$ will never equal 0 because it is always positive. So, since there exists an element of the second set that is not mapped from the first, β is not onto, as claimed.//

5.3.6. Let $A = \mathbf{R} - \{3\}$ and $B = \mathbf{R} - \{2\}$ and define $\gamma : A \to B$ by $\gamma(x) = \dfrac{2x+1}{x-3}$.

(a) Verify that γ maps A to B; that is, show that for all $a \in A$, $\gamma(a) \neq 2$. [HINT: Use contradiction.]

(b) Prove that γ is $1 - 1$.

(c) Prove that γ maps A onto B.

(a) (by contradiction) Let $\gamma(a)=2=\frac{2a+1}{a-3}$. Then 2a-6=2a+1 and 2a=2a+7. Clearly no such value of a exists, so γ does map A→B because ∀a∈A, γ(a)∈B.

(b) Proof: γ is one to one.

Show $(\forall x_1, x_2 \in A)\ \gamma(x_1) = \gamma(x_2) \to x_1 = x_2$. Let $\gamma(x_1) = \gamma(x_2)$. Then,

$$\frac{2x_1 + 1}{x_1 - 3} = \frac{2x_2 + 1}{x_2 - 3}$$

And multiplying across yields

$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$

Multiplying out and adding 3 to both sides yields

$$2x_1x_2 - 6x_1 + x_2 = 2x_1x_2 - 6x_2 + x_1$$

Taking out the common $2x_1x_2$ and putting x_1 on one side by itself leaves

$$7x_1 = 7x_2$$

As claimed.//

(c) Proof: γ maps A onto B.

Show ∀b∈B ∃a∈A s.t. γ(a)=b. Let z∈B. Set $x = \frac{-3z-1}{-z+2}$. Then

$$\gamma(x) = \frac{2\left(\frac{-3z-1}{-z+2}\right)+1}{\frac{-3z-1}{-z+2}-3}$$

Taking a common denominator for +1 and -3 on the top and bottom as –z+2:

$$\gamma(x) = \frac{\frac{-3z-1}{-z+2}+1(\frac{-z+2}{-z+2})}{\frac{-3z-1}{-z+2}-3(\frac{-z+2}{-z+2})} = \frac{\frac{-3z-1+(-z+2)}{-z+2}}{\frac{-3z-1-3(-z+2)}{-z+2}} = \frac{\frac{-7z}{-z+2}}{\frac{-7}{-z+2}} = \frac{-7z}{-z+2} \times \frac{-z+2}{-7}$$

The –z+2 terms and -7's cancel,

$$\gamma(x) = \frac{-7z}{-z+2} \times \frac{-z+2}{-7} = z.$$

Therefore, γ maps A onto B.//

Follow all that? (I guess that I do, I wrote it haha)